

On the exact R charge for $\mathcal{N} = 2$ CS theories

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ABSTRACT: Recently it was argued that the exact R charge for three dimensional $\mathcal{N} = 2$ supersymmetric field theories extremizes the partition function localized on S^3 . In this paper we check this conjecture by computing the R charge for $SU(N)_k$ YM CS gauge theories at large k for many representations, and we test the agreement with the perturbative results.

KEYWORDS: Chern-Simons Theories, Supersymmetric gauge theory, Matrix Models.

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1. Introduction

In the IR fixed point of a superconformal field theory the $U(1)_R$ charge usually differs from its UV value. Indeed the UV R_0 charge mixes with the global non-R abelian flavor symmetries F_i during the RG flow. At the fixed point the exact R charge that belongs to the superconformal algebra is a combination of these charges. In some cases, like $\mathcal{N} = 1$ 4d SQCD in the conformal window, the constraint arising from the superpotential and the requirement of anomaly cancellations determine the exact R charge at the IR fixed point. Anyway in many other cases these constraints are not enough and further informations are necessary. The problem of finding the exact IR R symmetry among all the trial combinations $R_t = R_0 + \alpha_i F_i$ has been solved for $\mathcal{N} = 1$ $d = 4$ superconformal field theories in [1]. It was shown that the combination of α_i that gives the exact R charge at the fixed point maximizes the conformal anomaly. This is associated with the Euler term in the trace anomaly of the stress tensor, $\langle T^\mu_\mu \rangle \simeq c(\text{Weyl})^2 - a(\text{Euler})$, where $a \simeq 3\text{Tr}R^3 - \text{Tr}R$. This result was shown by using the condition that the global $U(1)_R$ is free of ABJ anomalies (or equivalently the condition that the exact NSVZ beta function [2] vanishes for

all the gauge groups) and that the $U(1)_R$ 't Hooft anomalies determine the central charges a and c [3]. By using the 't Hooft anomaly matching it was indeed possible to compute these central charges in terms of the weakly coupled UV theory, and not from the IR physics, usually strongly coupled.

The same problem of finding the exact R charge at the superconformal fixed point can be formulated in $\mathcal{N} = 2$ $d = 3$. For higher amount of supersymmetry the $U(1)_R$ symmetry does not mix with the other abelian symmetries of the theory, and $R_{IR} = R_{UV}$. Unfortunately in the $\mathcal{N} = 2$ case the a -maximization cannot be performed, because the trace anomaly of the stress tensor vanishes in three dimensions. In [4] it was argued that in d dimensional superconformal gauge theories the exact R charge minimizes the coefficient τ of the R -current two point function

$$\langle J_R^\mu(x) J_R^\nu(y) \rangle \simeq \tau_{RR} (\partial^2 \delta^{\mu\nu} - \partial^\mu \partial^\nu) \frac{1}{(x-y)^{2(d-2)}}. \quad (1.1)$$

Anyway in three dimensional theories this quantity receives quantum corrections and a perturbative analysis is necessary.

Recently a different approach to this problem has been proposed in [5]. It was argued that in three dimensions the role of the central charge a is played by the three sphere partition function \mathcal{Z} . This procedure of putting the theory on a sphere, which is called localization, was already applied on $d = 4$ $\mathcal{N} = 2$ supersymmetric gauge theories in [6]. It consists of adding some Q -exact operator (where Q represent the supersymmetry generator on the sphere), such that the path integral does not change but the one loop determinants are exact. This is the key point of the computation. Indeed even if the R charge mixes with the abelian flavor symmetries, the partition function at one loop contains all of the information of this mixing. Recently in $d = 3$ the localized partition functions on S^3 have been investigated for $\mathcal{N} \geq 3$ supersymmetry [7,8], where it was show that the $N^{3/2}$ entropy scaling expected from the gravitational side of the $\text{AdS}_4/\text{CFT}_3$ correspondence [9] can be computed in field theory. Moreover the study of the $\mathcal{N} = 2$ case was started in [10], in theories where there are not quantum corrections. The novelty of [5] is that the quantum corrected partition function computed on S^3 is extremized by the exact IR R symmetry. In [11–13] this conjecture was strengthened by computing the partition function of some $\mathcal{N} = 2$ CS toric quiver gauge theories. Indeed in that case it is possible to compute the exact field theory R -charge by minimizing the volume of the Sasaki Einstein manifold [14]. Actually in four dimensions this function is matched with the conformal anomaly a even before the extremization [15]. The authors showed that a similar relations between the Sasaki Einstein volume and the partition function holds in three dimensions at large N .

In this paper we study the conjectured \mathcal{Z} extremization in CS matter theories at large CS level k . In [5] the $U(1)_k$ and $SU(2)_k$ CS gauge theories with flavor were discussed, and the agreement with the field theory perturbative analysis was checked.

The localization procedure reduces the path integral to a matrix integral over the Cartan subalgebra of the Lie algebra the $U(1)_k$ and $SU(2)_k$ and the integrals are over a single variable. Here we check the conjecture by computing the partition function of non abelian CS theories with higher dimensional Cartan subalgebra, with matter fields in the $R + \overline{R}$ representation. We study $SU(N)_k$ theories and we see that the computation agree with the perturbative results discussed in field theory. For large k the matrix integral are dominated by a gaussian factor $e^{k\text{Tr}u^2}$, where u_i are the eigenvalues in the Cartan basis. The computations is performed by expanding the partition function at the saddle point, $u \sim 0$. This expansion simplifies the integral by giving an overall Vandermonde determinant coming from the one loop determinant of the vector multiplet. By writing this Vandermonde determinant in terms of orthogonal polynomials we have been able to compute the exact partition function at the lowest order in the 't Hooft coupling.

In section 2 we review the basic aspect of the partition function. In section 3 we review the field theory results. In section 4 explain the strategy of the computation of the partition function for the $SU(N)_k$ theories, and we check that the R charge evaluated with this method agrees with the two loop results for fundamental, adjoint, symmetric and antisymmetric representations. Then we conclude. In appendix A we review the conventions we used on Lie algebra, while in appendix B we summarize some results for the integration of the orthogonal polynomials.

2. The partition function localized on S^3

As explained in the introduction the exact R charge is computed by extremizing the partition function of the localized $\mathcal{N} = 2$ action on S^3 of a YM CS theory with matter fields in the R representation of the gauge group. The computation of the partition function on S^3 follows from the localization techniques of [16]. This technique was first used by [6] for the computation of the partition function of S^4 for $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions. In three dimensions the first computation was performed in $\mathcal{N} \geq 3$ by [10]. Then in [5, 17] this computation was extended to the $\mathcal{N} = 2$ case, in which the partition function explicitly depends on the R charge, which is not trivial, because the R -symmetry is abelian, and some regularization is necessary. In the case of a YM-CS in $\mathcal{N} = 2$ three dimensional models they found that the relevant contribution to the partition function is

$$\mathcal{Z} \simeq \int \prod_{i=1}^N du_i e^{\pi i k \text{Tr}_F u^2} \det_{\text{adj}}(\sinh(\pi u)) \det_R e^{l(1-\Delta+iu)} \quad (2.1)$$

The different terms in (2.1) are

- the rk_G integration variables over the gauge group G . They are written in terms of the eigenvalues u_i which correspond to the weight of the fundamental

representation. These variables correspond to the scalars σ in the vector multiplet. They transform in the adjoint representation of the gauge group, and this allows the integrals to be taken over \mathbb{R} .

- the determinant over the adjoint is explicitly $\prod_{i < j} \sinh^2(\pi \rho_{ij}(u))$ and it is related to the one loop contribution of every vector multiplet¹. The weights $\rho_{ij}(u)$ correspond to the roots of the algebra because the vector multiplet is in the adjoint representation;
- the term $e^{\pi i k \text{Tr}_F u^2}$ is the contribution of the k -level CS term. The sum is over the fundamental representation and it explicitly becomes $\text{Tr}_F u^2 = \sum_i u_i^2$. On the contrary the YM term does not contribute to the partition function;
- the one loop contribution of the matter fields is $\det_R e^{l(1-\Delta+iu)}$. The determinant is over the representation. Explicitly it is

$$\det_R e^{l(1-\Delta+iu)} = \prod_R e^{l(1-\Delta+i\rho_i(u))} \quad (2.2)$$

where ρ are the weights of R in terms of the eigenvalues u_i . The function $l(z)$ was introduced in [5] (see also [17] for an independent derivation). It arises from the regularization of the one loop determinant of the matter fields. This one loop determinant is

$$\prod_{n=1}^{\infty} \left(\frac{n+1-\Delta+i\rho(u)}{n+1-\Delta-i\rho(u)} \right)^n \quad (2.3)$$

If there are not quantum corrections to the conformal dimension Δ the classical value $\Delta = 1/2$ simplifies this expression and the regularized one loop determinant is [10]

$$\frac{1}{\cosh \rho_i(u)} \quad (2.4)$$

Anyway in this case we expect the R charge to couple with other flavor symmetry, and the classical result is quantum corrected. By using the Zeta functions the regularized function for the one loop determinant is $e^{l(z)}$ where

$$l(z) = -\frac{i\pi}{12} - z \text{Log}(1 - e^{2i\pi z}) + \frac{1}{2}i \left(\pi z^2 + \frac{Li_2(e^{2i\pi z})}{\pi} \right) \quad (2.5)$$

and $z = 1 - \Delta + i\rho_i(u)$.

In this paper we restrict to the $SU(N)_k$ cases in the $R + \overline{R}$, with R fundamental, adjoint, symmetric or antisymmetric. In appendix A we give some useful details for the weights of these representations.

¹We are omitting a factor 2, which is overall and does not affect the computation of the R charge.

The claim made in [5] is that this partition function is extremized by the exact R charge. This result is based on the assumption that the partition function depends holomorphically on the combination $\Delta_j - im_j$, where m_j is a real mass term for the j -th chiral multiplet, which conformal dimension is Δ_j . From this holomorphy it follows that $\partial_\Delta \mathcal{Z} \simeq \partial_m \mathcal{Z}$. The one point function of an operator in a CFT over S^3 is $\frac{1}{\mathcal{Z}} \partial_m \mathcal{Z}|_{m=0, \Delta=\Delta_{IR}}$, and it vanishes at the conformal fixed point if parity is preserved. If parity is broken the 1-point function can be proportional to the identity. The identity is a parity invariant operator, its VEV is real and $\text{Im}(1/\mathcal{Z} \partial_m \mathcal{Z})$ vanishes. The result is that in this case the exact R charge extremizes $|\mathcal{Z}|^2$.

In [5] this conjecture has been checked for cases with a one dimensional Cartan subalgebra. It was shown that R charge computed from the partition function agrees with the perturbative result of [18].

Here we check the conjecture for $SU(N)_k$ CS matter theories at large k , and we observe that the R charge computed from the partition function agrees to the field theory expected result even if the Cartan subalgebra has dimension higher than one.

3. Field theory result

Here we briefly review the basic aspect of the field theory that we are considering, discussed in [18]. The theory is a $\mathcal{N} = 2$ CS theory coupled with matter fields. The vector superfield V is composed by the gauge field A_μ , the auxiliary scalars σ and D and the two component Dirac spinor χ . The CS term in the action (in the WZ gauge) is

$$S_{CS} = \frac{k}{4\pi} \int \text{Tr} \left(A \wedge dA + \frac{2}{3} A^3 - \chi \bar{\chi} + 2D\sigma \right) \quad (3.1)$$

where k is an integer denoting the CS level. The matter field $\Phi = (\phi, \psi)$ is coupled to this vector multiplet by the action

$$S_{\text{matter}} = \int D_\mu \bar{\phi} D^\mu \phi + i \bar{\psi} \gamma^\mu D_\mu \psi - \bar{\phi} \sigma^2 \phi + \bar{\phi} D \phi - \bar{\psi} \sigma \psi + i \bar{\phi} \bar{\chi} \psi - i \bar{\psi} \chi \phi \quad (3.2)$$

By integrating out χ and D the authors observed that this action is not only classically but also quantum mechanically marginal. By adding a $\mathcal{N} = 3$ coupling of Φ with Q and \tilde{Q} , in the $R + \bar{R}$ of the gauge group, they showed that the model reduces to a WZ model at large k . In the $\mathcal{N} = 2$ case the coefficient of the two loops beta function is

$$b_0(R + \bar{R}) = \frac{2}{|R|} \left((\text{Tr} T^a T^b)^2 + \text{Tr} T^a T^b T^a T^b \right) \quad (3.3)$$

and the R charge for these matter fields is

$$R_{Q, \tilde{Q}} = \frac{1}{2} - \frac{b_0}{4k^2} + \mathcal{O} \left(\frac{1}{k^4} \right) \quad (3.4)$$

where we are normalizing the trace of the fundamental such that $T(F) = 1$. The traces appearing in (3.3) are

$$\begin{aligned}\mathrm{Tr} T^a T^b \mathrm{Tr} T^a T^b &= |G| T(r)^2 \\ \mathrm{Tr} T^a T^b T^a T^b &= |G| T(r) \left(\frac{T(r)|G|}{|r|} - \frac{T(G)}{2} \right)\end{aligned}\quad (3.5)$$

In the case of N_f flavor in the fundamental and antifundamental of $SU(N)_k$ we have

$$R_{SU(N)_k}(Q, \tilde{Q}) = 1 - \frac{(N_f N - 1)(N^2 - 1)}{2k^2 N^2} \quad (3.6)$$

In the next sections we will show that the same result is obtained by extremizing the partition function.

4. Exact R in $\mathcal{N} = 2$ $SU(N)_k$ CS matter theories

In this section we explain general aspects of the computation of the exact R charge from the extremization of \mathcal{Z} . We concentrate on representation $R + \overline{R}$, at large k , because we want to compare with the perturbative field theory results. By restricting to large values for the CS level k the matrix integral (2.1) is dominated by the saddle point $u_i = 0$. In the $SU(N)$ case there are $N - 1$ independent variables u_i , because of the traceless condition. The gaussian measure is

$$\mathrm{Exp} \left(2i\pi k \left(\sum_{i=1}^N u_i^2 + \sum_{i < j} u_i u_j \right) \right) \quad (4.1)$$

It is possible to operate a coordinate transformation on the eigenvalues to diagonalize this term, but this procedure complicates the one loop contributions of the vector and the matter fields in terms of the eigenvalues. We followed a different strategy: we integrate over N and not $N - 1$ eigenvalues, by imposing the traceless condition on the weights, $\mathrm{Tr} u = 0$. We show this procedure with an example. If we consider the gaussian integral

$$\int_R e^{-2\alpha x^2} dx = \sqrt{\frac{\pi}{2\alpha}} \quad (4.2)$$

we can write this as

$$\int_R \int_R e^{-2\alpha(x^2+y^2)} \delta(x+y) = \int_R \int_R e^{-2\alpha(x^2+y^2)} \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} e^{im(x+y)} \quad (4.3)$$

where we used the Fourier expansion for the delta function². The gaussian integrations on x and y can be separately performed by completing the squares in the

²Alternatively one can use the Fourier transform of the δ function and observe that the two results agree because the integral receives contributions only at $u_i \simeq 0$.

exponentials. By comparing the integral in (4.2) with the one in (4.3) we obtain

$$S_0 \equiv \sum_{m=-\infty}^{\infty} e^{-\frac{m^2}{2\alpha}} = \sqrt{2\pi\alpha} \quad (4.4)$$

We will use this procedure to compute the partition function for the $SU(N)$ gauge theory, by imposing the constraint with a delta function. Moreover we will explicitly use the series S_0 and his derivative to respect to α . The partition function becomes

$$\mathcal{Z} = \sum_{m=-\infty}^{\infty} e^{-\frac{iNm^2}{4k\pi}} \int \prod_{i=1}^N du_i e^{i\pi k(u_i + \frac{m}{2k\pi})^2} \prod_{i<j} \sinh^2(u_i - u_j) \det_R e^{N_f(l(1+\Delta+i\rho(u))+l(1+\Delta-i\rho(u)))} \quad (4.5)$$

Observe that the restriction to $R+\bar{R}$ simplifies the weight dependence of the partition function. Indeed here we use the relation $\bar{\rho}(u) = -\rho(u)$, i.e. the weights of the conjugate representation \bar{R} are opposite to the weights of R .

Then we shift the eigenvalues by a factor $\frac{m}{2\pi k}$. This shift does not affect the one loop contribution of the vector field. By expanding this term around $u_i \sim 0$ we have

$$\begin{aligned} \Delta^2(\sinh(\pi u)) &\simeq \Delta^2(\pi u) \prod_{i<j} \left(1 + \frac{\pi^2}{3}(u_i - u_j)^2 + \frac{2\pi^4}{45}(u_i - u_j)^4 \right) \\ &\simeq \Delta^2(\pi u) \left(1 + \frac{\pi^2}{3} \sum_{i<j} (u_i - u_j)^2 - \frac{\pi^4}{90} \sum_{i<j} (u_i - u_j)^4 + \frac{\pi^4}{18} \left(\sum_{i<j} (u_i - u_j)^2 \right)^2 \right) \end{aligned} \quad (4.6)$$

where we used the same strategy of [19, 20], computing the partition functions of matrix integrals with $\sinh^2 \Delta$ kernel. Formula (4.6) can be written in a more suitable way to perform the integration over the gauge group. The final result that we get is

$$\begin{aligned} \Delta^2(\sinh(\pi u)) &\simeq \Delta^2(\pi u) \left(1 - \frac{\pi^2(N-1)}{3} \sum_{i=1}^N u_i^2 - \frac{2\pi^2}{3} \sum_{i<j} u_i u_j \right. \\ &+ \frac{\pi^4(5N^2-11N+6)}{90} \sum_i u_i^4 - \frac{2\pi^4(5N^2-6)}{45} \sum_{i<j} (u_i^3 u_j + u_i u_j^3) - \frac{2\pi^2(N-3)}{9} \sum_{j<k} u_i^2 u_j u_k \\ &\left. + \frac{\pi^4(5N^2-10N+12)}{45} \sum_{i<j} u_i^2 u_j^2 + \frac{4\pi^4}{3} \sum u_i u_j u_l u_k \right) \end{aligned} \quad (4.7)$$

All the terms in the parenthesis are integrated over du_i by writing the Vandermonde determinant in terms of Hermite polynomials, orthogonal to respect to the measure

³Here we fix a convention which holds in all the paper. When a term like $u_{i_A} u_{i_B}$ appears in the sum, if we don't specify any condition (like $i_A < i_B$), the constraint $i_A \neq i_B$ is understood. For example the term $u_i u_j u_k u_l$ refers to the combination of four different weights.

$e^{\pi i k u^2} du$ [21]. These polynomials are

$$p_n(u) = \left(\frac{1}{i\pi k} \right)^n e^{-\frac{i\pi k u^2}{2}} \frac{d^n}{du^n} e^{\frac{i\pi k u^2}{2}} \quad (4.8)$$

normalized such that $h_n = \langle p_n, p_n \rangle$ is

$$h_n = \frac{1}{\sqrt{2}\pi^{n+1}} \left(-\frac{i}{k} \right)^{n+\frac{1}{2}} n! \quad (4.9)$$

The Vandermonde determinant in term of these polynomials is $\Delta(\pi u) = \det(p_{N-1}(u))$ or explicitly

$$\begin{aligned} \Delta(\pi u) &= \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ \pi u_1 & \pi u_2 & \dots & \pi u_N \\ \dots & \dots & \dots & \dots \\ (\pi u_1)^{N-1} & (\pi u_2)^{N-1} & \dots & (\pi u_N)^{N-1} \end{pmatrix} \\ &= \det \begin{pmatrix} p_0(u_1) & p_0(u_2) & \dots & p_0(u_N) \\ p_1(u_1) & p_1(u_2) & \dots & p_1(u_N) \\ \dots & \dots & \dots & \dots \\ p_{N-1}(u_1) & p_{N-1}(u_2) & \dots & p_{N-1}(u_N) \end{pmatrix} \end{aligned} \quad (4.10)$$

The integration is performed by applying the recursion relation of the normalized polynomials $P_n(u) = p_n(u)/\sqrt{h_n}$,

$$uP_n(u) = \sqrt{r_{n+1}}P_{n+1}(u) + \sqrt{r_n}P_{n-1}(u) \quad (4.11)$$

where $r_n = h_n/h_{n-1} = n/(i\pi k)$. In the appendix B we list all the integrals necessary to complete the computation. After the expansion of the Vandermonde we have to expand the 1-loop determinant contribution of the matter fields. In the large k limit we expand this term around $\Delta = 1/2$ because we expect a small deviation from the classical result. From now on we define $\Delta = 1/2 + a$ and expand at $a \sim 0$.

The shift on the eigenvalues modifies the function $l(z)$. Indeed in a generic representation a shift of a constant value ξ on the eigenvalues modifies the weights from $\rho(u)$ to $\rho(u + \xi)$ (an exception is the adjoint representation, as we already observed). The effect of the shift becomes

$$\rho(u) \rightarrow \rho(u + \xi) = \rho(u) + T\xi \quad (4.12)$$

because the weights are linear functions of $\rho(u)$; T represents the number of boxes in the Young tableaux associated to the representation that we are considering, or equivalently the number of u_i in the weights. In our case $\xi = \frac{m}{2\pi k}$. The one loop

determinant is eventually expanded for large k and for $u_i \sim 0$ and $a \sim 0$. This expansion is formally written as

$$\prod_{i=1}^N e^{N_f(l(1-\Delta+i(u_i-\frac{m}{2k\pi}))+l(1-\Delta-i(u_i-\frac{m}{2k\pi})))} = \sum_{i,j} \sum_{\alpha,\beta} f_{\alpha\beta\gamma}(a) u_i^\alpha u_j^\beta u_{(m)}^\gamma \quad (4.13)$$

The u_i dependence is determined by the knowledge of $\sum \rho_i^2$ and $\sum \rho_i^4$, where the sums run only over the weights of R , if the representation is $R + \overline{R}$. Moreover we defined $u_{(m)} = m/k$, because this term has a similar role than u_i in the large k expansion. Indeed the u_m expansion adds some powers on m^2 , which modify some of the series over m . The modified series are

$$S_2 = \sum_{m=-\infty}^{\infty} \frac{m^2}{k^2} e^{-\frac{iNm^2}{4k\pi}} = -\frac{4\pi i}{N} \partial_k S_0, \quad S_4 = \sum_{m=-\infty}^{\infty} \frac{m^4}{k^4} e^{-\frac{iNm^2}{4k\pi}} = -\frac{16\pi^2}{N} \left(\partial_k^2 + \frac{2}{k} \partial_k \right) S_0 \quad (4.14)$$

By computing the relevant terms in the product of (4.7) and (4.13) and by using the integrals in the appendix B and the series S_0, S_2 and S_4 we obtain the partition function. The last step for finding the exact R charge is then solving the equation $\partial_a |Z|^2 = 0$. In the rest of this section we apply this procedure to many representation of $R + \overline{R}$, showing the agreement of this technique and the expected perturbative results.

4.1 Fundamental and antifundamental representations

Here we compute the partition function for a generic $SU(N)_k$ gauge theory with $N \ll k$ and with N_f fundamental and N_f antifundamental quarks⁴. The partition function is

$$\mathcal{Z} = \int \prod_{i=1}^N du_i e^{i\pi k u_i^2 + N_f(l(1-\Delta+iu_i)+l(1-\Delta-iu_i))} \prod_{i<j} \sinh^2(u_i - u_j) \delta(u_1 + \dots + u_n) \quad (4.15)$$

As we anticipated above we write the delta function in terms of his Fourier series and then we make a change of variables, by completing the squares in the exponential. The partition function that we obtain is

$$\mathcal{Z} = \sum_{m=-\infty}^{\infty} e^{-\frac{iNm^2}{4k^2\pi}} \int \prod_{i=1}^N du_i e^{i\pi k u_i^2 + N_f(l(1-\Delta+i(u_i-\frac{m}{2k\pi}))+l(1-\Delta-i(u_i-\frac{m}{2k\pi})))} \prod_{i<j} \sinh^2(u_i - u_j) \quad (4.16)$$

We then factor out the Vandermonde determinant $\Delta^2(\pi u)$ as in (4.7), and we write it in terms of orthogonal Hermite polynomials. The 1 loop determinants are expanded

⁴In [5] the author used a different convention, $N_f^{his} = N N_f^{our}$.

as in (4.13). Explicitly the weights are $\rho_i = u_i$ and the u_i dependence of the expansion is

$$\sum_{Fund.} \rho^2 = \sum_{i=1}^N u_i^2 \quad , \quad \sum_{Fund.} \rho^4 = \sum_{i=1}^N u_i^4 \quad (4.17)$$

The partition function at the leading order in a at large k becomes

$$\begin{aligned} \mathcal{Z} \simeq & 288k^2N(2 + a^2N_fN\pi^2) + 24i\pi kN(N^2 - 1)(4N + N_f(a(24 + a(N^2 - 3N_fN - 6)\pi^2) - 6)) \\ & - (N^2 - 1)\pi^2(8N^5 + 18(4a - 1)^2N_f^2N(N^2 + 1) - 12N_f(3 + 2N^4 - 8a(N^4 - N^2 + 3)) - a^2N_f(144 \\ & - N(9N_f^2N(N^2 + 1) + 6N_f(2N^4 - 6N^2 - 3) - 4N(N^4 - 6N^2 + 18)))\pi^2) \end{aligned} \quad (4.18)$$

where we omitted the overall factors, which do not modify the extremization. By extremizing $|\mathcal{Z}|^2$ to respect to a we finally obtain

$$R = \Delta = \frac{1}{2} - a = \frac{1}{2} - \frac{(N_fN - 1)(N^2 - 1)}{2k^2N^2} \quad (4.19)$$

exactly as expected from the perturbative computation.

4.2 Adjoint representation

Here we discuss the case of matter fields in the adjoint representation. By observing that the adjoint is self conjugate $R_{Adj} = R_{\overline{Adj}}$ the g adjoints fields can be thought as $g/2$ fields and $g/2$ conjugate fields⁵. By applying this observation to the field theory computation we expect that for g adjoint the R charge is

$$R = 1 - \frac{N^2(g + 1)}{k^2} \quad . \quad (4.20)$$

In this case the shift introduced by the delta function does not affect the integrations because the weights appearing in the one loop determinant are in the adjoint representation. Indeed the shift

$$u_i \rightarrow u_i - \frac{m}{2\pi k} \quad (4.21)$$

gives an overall factor, but cancels in the differences $u_i - u_j$.

The rk_G zero-weights of the adjoint contribute to the partition function. In $SU(N)_k$ they are $N - 1$ and they add to the 1-loop determinant a factor $\text{rk}_G \times l(1 - \Delta)$. The partition function for the $SU(N)_k$ gauge theory with g adjoint matter fields is

$$\mathcal{Z} \simeq \int \prod_{i=1}^N du_i e^{i\pi k u_i^2} \prod_{i < j} \sinh^2(\pi(u_{ij})) \left(e^{g(l(1 - \Delta + iu_{ij}) + l(1 - \Delta - iu_{ij}))} \right) e^{g(N - 1)l(1 - \Delta)} \quad (4.22)$$

⁵We restrict to an even number of adjoint, such that we can use the formula (3.3) as explained in [18].

where $u_{ij} = u_i - u_j$, and we omitted the overall series introduced by the delta function. The one loop determinant for the vector field is expanded as above. Moreover the weights $\rho(u) = u_i - u_j$ contribute to the one loop determinant of the matter field as

$$\sum_{Adj.} \rho^2(u) = (N-1) \sum_{i=1}^N u_i^2 - 2 \sum_{i<j} u_i u_j \quad (4.23)$$

and

$$\sum_{Adj.} \rho^4(u) = (N-1) \sum_{i=1}^N u_i^4 + 6 \sum_{i<j} u_i^2 u_j^2 + 4 \sum_{i<j} (u_i^3 u_j + u_i u_j^3) \quad (4.24)$$

The partition function is proportional to

$$\begin{aligned} & 288k^2(4 + a^2g(N^2 - 1)\pi^2) + 24i\pi kN(N^2 - 1)(8 + g(a(48 + a(3g(1 - N^2) + 2N^2 - 14)\pi^2) - 12)) \\ & - \pi^2 N^2(N^2 - 1)(12g(6 - 64a + 3(4a - 1)^2g) + 4(2 + 3(4a - 1)g)^2N^2 + a^2g(9(48 - (g - 6)g) + 2(51g \\ & - 26)N^2 + (3g - 2)^2N^4)\pi^2) \end{aligned} \quad (4.25)$$

where we omitted the overall factor. The R -charge obtained from the extremization of this partition function coincides with (4.20).

4.3 Symmetric representation

Here we consider a matter field in the M -index symmetric representation and its conjugate. The weights are

$$\rho_i(u) = u_{i_1} + \cdots + u_{i_M} \quad (4.26)$$

without any restrictions on the u_i . We compute the sums $\sum \rho^2$ and $\sum \rho^4$ of the weights in terms of the eigenvalues u_i . By defining $C_i^j = \binom{i}{j}$ we obtained

$$\sum_{Sym.} \rho^2 = \sum_{i=0}^M (M-i)^2 C_{N+i-2}^i \sum_{i=1}^N u_i^2 + 2 \sum_{i=1}^{M-1} (M-i) C_{N+i-2}^{i-1} \sum_{i<j} u_i u_j \quad (4.27)$$

where the coefficients in the sum over i can be written in terms of the dimension of the representation $|R|$ as

$$\sum_{i=0}^M (M-i)^2 C_{N+i-2}^i = \frac{M(N+2M-1)}{N(N+1)} |R|, \quad \sum_{i=1}^{M-1} (M-i) C_{N+i-2}^{i-1} = \frac{M(M-1)}{N(N+1)} |R| \quad (4.28)$$

The expansion of $\sum \rho^4$ is

$$\begin{aligned}
\sum_{Sym.} \rho^4 &= \sum_{i=0}^M (M-i)^4 C_{N+i-2}^i \sum_{i=1}^N u_i^4 + 4 \sum_{i=1}^{M-1} (M-i)^3 C_{N+i-2}^{i-1} \sum_{i < j} (u_i^3 u_j + u_i u_j^3) \\
&+ 6 \sum_{i=1}^{M-1} (M-i)^2 (C_{N+i-2}^{i-1} + 2C_{N+i-2}^{i-2}) \sum_{i < j} u_i^2 u_j^2 + 12 \sum_{i=2}^{M-2} (M-i)^2 C_{N+i-2}^{i-2} \sum_{j < k} u_i^2 u_j u_k \\
&+ 24 \sum_{i=3}^{M-3} (M-i) C_{N+i-2}^{i-3} \sum u_i u_j u_l u_k
\end{aligned} \tag{4.29}$$

Even in this case the coefficients can be written in terms of $|R|$. The partition function is proportional to

$$\begin{aligned}
&288k^2 N^2 (a^2 |R| \pi^2 + 2)(N+1) + 24ik\pi N(N^2-1)(4N^2(N+1) + 6(4a-1)|R|M(M+N) - a^2 |R| \\
&(3(|R|+2)(M+N)M - 2N^2(N+1))\pi^2) - (N-1)\pi^2(8N^6(N+1)^2 + 9a^2 |R|^3 M^2(M+N)^2(N^2+1)\pi^2 \\
&- 4|R|(N+1)(3M(M+N)(3(8a-1)M(N-1)(M+N) + N^2(3+2N^2-8a(N^2+2))) + a^2(6M^2(N-3)^2 \\
&N^2 - N^6(N+1) - 36M^3(N-1)(M+2N)6MN^3(N^2+3))\pi^2) + 6|R|^2 M(M+N)(3(4a-1)^2 M(M+N) \\
&(N^2+1) + a^2(3M(M+N)(3N^2+1) - N^2(N+1)(2N^2+3))\pi^2))
\end{aligned} \tag{4.30}$$

where we are omitting the overall term. By extremizing $|\mathcal{Z}|^2$ to respect to a we found

$$R = \frac{1}{2} - a = \frac{1}{2} - \frac{M(N-1)(M+N)((N+1)M^2 + M(N+1)(|R| + |G|))}{2k^2 N^2(N+1)} \tag{4.31}$$

By observing that for a M -index symmetric representation

$$T(r) = \frac{M(N+M)}{N(N+1)} |R| \tag{4.32}$$

our result agrees with the two loop field theory computation.

4.4 Antisymmetric representation

Here we compute the partition function for matter fields in the L index antisymmetric representation and its conjugate $\bar{L} = N - L$. The weights $\rho(u)$ are

$$\rho(u) = u_{i_1} + \cdots + u_{i_L} \quad \text{with} \quad i_j \neq i_k \quad \text{if} \quad j \neq k \tag{4.33}$$

The u_i dependence of the one loop determinant for the antisymmetric matter field is computed from the relation

$$\sum_{Antisym.} \rho^2(u) = C_{N-1}^{L-1} \sum u_i^2 + 2C_{N-2}^{L-2} \sum_{i < j} u_i u_j \tag{4.34}$$

and

$$\begin{aligned}
\sum_{\text{Antisym.}} \rho^4(u) &= C_{N-1}^{L-1} \sum u_i^4 + 4 C_{N-2}^{L-2} \sum_{i < j} (u_i^3 u_j + u_i u_j^3) + 6 C_{N-2}^{L-2} \sum_{i < j} u_i^2 u_j^2 \\
&+ 12 C_{N-3}^{L-3} \sum_{j < k} u_i^2 u_j u_k + 24 C_{N-4}^{L-4} \sum u_i u_j u_k u_l
\end{aligned} \tag{4.35}$$

Even in this case we can write the binomial coefficients in terms of the dimension of the representation. We then computed the partition function, it is proportional to

$$\begin{aligned}
&288k^2 N^2 (a^2 |R| \pi^2 + 2)(N-1) + 24ik\pi N(N^2-1)(4(N-1)N^2 + 6(4a-1)|R|L(N-L) - a^2|R| \\
&(3(|R|+2)L(N-L) - 2(N-1)N^2\pi^2) - (N+1)\pi^2(8(N-1)^2N^6 + 9a^2|R|^3 L^2(N-L)^2(N^2+1)\pi^2 \\
&- 4|R|(N-1)(3L(N-L)(3(8a-1)L(N-L)(N+1) + N^2(2N^2+3-8a(N^2+2)))) + a^2((N-1)N^6 \\
&- 6L^2N^2(N+3)^2 - 6LN^3(N^2+3)\pi^2) + 6|R|^2L(N-L)(3(4a-1)^2L(N-L)(N^2+1) - a^2((N-1) \\
&N^2(2N^2+3) + 3L(N-L)(3N^2+1)\pi^2))
\end{aligned} \tag{4.36}$$

By extremizing $|\mathcal{Z}|^2$ to respect to a we found

$$R = \frac{1}{2} - a = \frac{1}{2} - \frac{L(N+1)(N-L)((N-1)L^2 - L(N-L)(|R| + |G|))}{2k^2 N^2(N-1)} \tag{4.37}$$

which agrees with the field theory computation once we observe that

$$T(r) = \frac{L(N-L)}{N(N-1)} |R| \tag{4.38}$$

5. Conclusions

In this paper we checked that the extremization of the partition function of $\mathcal{N} = 2$ three dimensional YM CS theories leads to the correct R charge, by matching the results of $SU(N)_k$ theories with the perturbative results. We explicitly computed the partition function and the R charge for the fundamental, adjoint, symmetric and antisymmetric representations. It would be interesting to check this agreement for other representations and gauge groups.

Another interesting problem is the application of this procedure to $\mathcal{N} = 2$ CS matter theories conjectured to be dual to M theory on $\text{AdS}_4 \times \text{SE}_7$, where SE_7 is a seven dimensional Sasaki Einstein manifold [22]. Indeed in that case the exact R symmetry is known from the minimization of the volumes. In [11] this study has been started, and the authors showed that at large N the results agree for some two and three gauge group theory. It would be interesting extend the computation to theories with more gauge groups and check if the partition functions coincide for the proposed Seiberg and toric dualities of [23].

A last observation concerns the existence of a c -theorem in three dimensions. In four dimensions the Cardy conjecture [24] argues that the central charge is the

analogous of the Virasoro central charge of two dimensional gauge theories [25]. It takes into account the reduction of the massless degrees of freedom of the theory between the UV and the IR fixed point. In its strongest version the conjectured a -theorem states that a is a positive monotonic decreasing function in an RG flow from the UV to the IR. Many checks of this conjecture exist, confirming the prediction of a monotonic decreasing behavior of a during the RG flow, by using some refined version of a -maximization out of the fixed point [26]. Here we observe that in all the examples we computed the extremized value of the R charge is a minimum of the partition function. By supposing that the actual procedure of finding the exact R charge is the minimization of the partition function, this last quantity may play the role of the central charge a (or its inverse) in four dimension, and a c -theorem may be formulated (see [13] for a recent discussion on this possibility).

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A. Conventions on Lie Algebra

In this appendix we review the conventions on Lie algebra that we used to perform the computation. We computed the partition functions in terms of the weights of the fundamental representation. The weights of the other representations are expressed in terms of these weights. We choose the following basis for the weights of the fundamental representation

$$u_1 = (1, 0, \dots, 0), \quad u_2 = (-1, 1, 0, \dots, 0), \quad \dots \quad u_N = (0, \dots, -1) \quad (\text{A.1})$$

with $N - 1$ elements in each weight. The weights of the antifundamental are the same with opposite sign. The highest weight, u_1 is associated to the A_{N-1} Dynkin diagram describing $SU(N)$. There is a correspondence between the highest weight and the representations, because the i -th element N_i in the highest weight is the number of column of length N_i in the Young tableaux. A generic representation is always described by its highest weight, and its knowledge is enough to construct all the other weights. These other weights are computed by acting with the rows of the Cartan matrix on the highest weight. Moreover it is possible to express the rows of the Cartan matrix to respect of the weights of the fundamental representation. Indeed they are simply $u_i - u_{i+1}$. By subtracting these quantities to the highest weight of a generic representation the other weights are computed. If we consider the adjoint representation its highest weight is $(1, 0, \dots, 0, 1)$, which in terms of the fundamental representation is $u_1 - u_n$, the action of $u_i - u_{i+1}$ gives the usual $u_i - u_j$

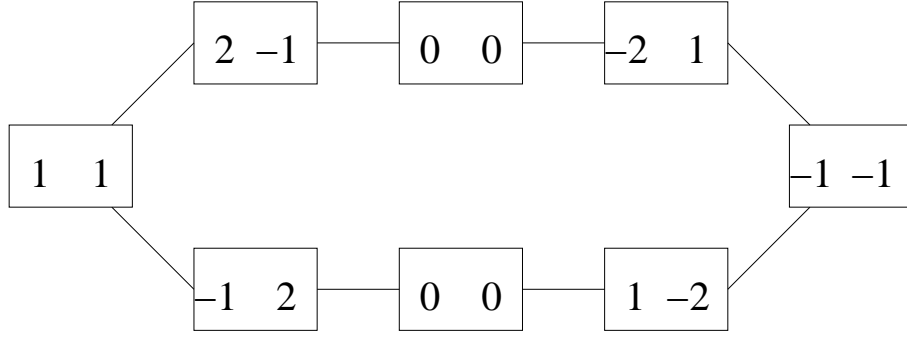


Figure 1: Weights of the Adjoint of $SU(3)$

$N(N-1)/2$ non zero weights and in addition there are $N-1$ zero weights. For example for $SU(3)$ the highest weight is $u_1 - u_3 = (1, 1)$. Then we subtract the rows $u_1 - u_2 = (2, -1)$ and $u_2 - u_3 = (-1, 2)$ obtaining the other weights. Usually it is not necessary to introduce this specific basis for the u_i and the weights of the representation are derived as in figure 1. Anyway here we choose the basis u_i because it is more convenient for the integrations over the gauge group. In the paper we studied also the M-index symmetric representation $(M, 0, \dots, 0)$ (with his conjugate $(0, \dots, M)$) and the L-index antisymmetric $(0, \dots, 1_L, \dots, 0)$ (with its conjugate). We computed the weights for these representations by starting from the highest one, and finally we expressed the sum $\sum \rho^2$ and $\sum \rho^4$ in term of (A.1).

B. Integrals

In this appendix we list the integrals that we used to perform the computation for the $U(N)_k$ theory. The generic integral is

$$I = \int \prod_i du_i e^{ik\pi \text{Tr} u^2} f(\{u_i\}) \Delta^2(\pi u) \quad (\text{B.1})$$

If $f(\{u_i\}) = 1$ we obtain

$$I_1 = \sqrt{(-1)^N} e^{-\frac{1}{4}iN^2\pi} k^{-\frac{N^2}{2}} \left(\frac{\pi}{2}\right)^{\frac{1}{2}(N-1)N} G_2(N+2) \quad (\text{B.2})$$

where G_2 is the Barnes function defined by $G_2(z+1) = \Gamma(z)G_2(z)$, and $G_2(1) = 1$. This integral differs from the usual integral of the Vandermonde by a power of $2^{N(N-1)}$, because we omitted a factor of 2 in front of $\sinh^2(\pi(u_i - u_j))$ in the original partition function. For different values of the function $f(\{u_i\})$ the integrals are computed from the recursive relation of the orthogonal polynomials. The integrals

that are necessary for our computation are listed below.

$f(\{u_i\})$	I/I_1
$\sum u_i^2$	$\frac{iN^2}{2\pi k}$
$\sum_{i<j} u_i u_j$	$-\frac{iN(N-1)}{4\pi k}$
$\sum_i u_i^4$	$-\frac{N(2N^2+1)}{4\pi^2 k^2}$
$\sum_{i<j} u_i^2 u_j^2$	$-\frac{N(N-1)(N^2-N+1)}{8\pi^2 k^2}$
$\sum_{j<k} u_i^2 u_j u_k$	$\frac{N(N-1)(N-2)^2}{8\pi^2 k^2}$
$\sum_{i<j} (u_i^3 u_j + u_j^3 u_i)$	$\frac{N(N-1)(2N-1)}{4\pi^2 k^2}$
$\sum u_i u_j u_k u_l$	$-\frac{N(N-1)(N-2)(N-3)}{32\pi^2 k^2}$

(B.3)

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